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Prasoon Kumar

MATHS OLYMPIAD

International Mathematics Olympiad

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INTERNATIONAL MATHEMATICS OLYMPIAD

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प्रकाशक

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पुस्तक में प्रदान की गयी सभी सामग्रियों को व्यावसायिक मार्गदर्शन के तहत सरल बनाया गया है। किसी भी प्रकार के उद्धरण या अतिरिक्त जानकारी के स्रोत के रूप में किसी संगठन या वेबसाइट के उल्लेखों का लेखक या प्रकाशक समर्थन नहीं करता है। यह भी संभव है कि पुस्तक के प्रकाशन के दौरान उद्धृत बेवसाइट हटा दी गयी हो।

इस पुस्तक में उल्लिखित विशेषज्ञ के राय का उपयोग करने का परिणाम लेखक और प्रकाशक के नियंत्रण से हटकर पाठक की परिस्थितियों और कारकों पर पूरी तरह निर्भर करेगा।

पुस्तक में दिये गये विचारों को आजमाने से पूर्व किसी विशेषज्ञ से सलाह लेना आवश्यक है। पाठक पुस्तक को पढ़ने से उत्पन्न कारकों के लिए पाठक स्वयं पूर्ण रूप से जिम्मेदार समझा जायेगा।

उचित मार्गदर्शन के लिए पुस्तक को माता-पिता एवं अभिभावक की निगरानी में पढ़ने की सलाह दी जाती है। इस पुस्तक के खरीददार स्वयं इसमें दिये गये सामग्रियों और जानकारी के उपयोग के लिए सम्पूर्ण जिम्मेदारी स्वीकार करते हैं।

इस पुस्तक की सम्पूर्ण सामग्री का कॉपीराइट लेखक/प्रकाशक के पास रहेगा। कवर डिजाइन, टेक्स्ट या चित्रों का किसी भी प्रकार का उल्लंघन किसी इकाई द्वारा किसी भी रूप में कानूनी कार्रवाई को आमंत्रित करेगा और इसके परिणामों के लिए जिम्मेदार समझा जायेगा।

Publisher's Note

V&S Publishers, after the grand success of a number of academic and general books, is pleased to bring out a series of *Mathematics Olympiad books* under *The Gen X series – generating Xcellence in generation X* – which has been designed to focus on the problems faced by students. In all books the concepts have been explained clearly through various examples, illustrations and diagrams wherever required. Each book has been developed to meet specific needs of students who aspire to get distinctions in the field of mathematics and want to become Olympiad champs at national and international levels.

To go through Maths Olympiad successfully, students need to do thorough study of topics covered in the *Olympiads syllabus and the topics covered in school syllabus as well*. The Olympiads not only tests the subjective knowledge but Reasoning skills also. So students are required to comprehend the depth of concepts and problems and gain experience through practice. The Olympiads check efficiency of candidates in problem solving. These exams are conducted in different stages at regional, national, and international levels. At each stage of the test, the candidate should be fully prepared to go through the exam. Therefore, this exam requires careful attention towards comprehension of concepts, thorough practice, and application of rules and concepts.

While other books in market focus selectively on questions or theory; V&S Maths Olympiad books are rather comprehensive. Each book has been divided into five sections namely *Mathematics, Logical Reasoning, Achiever's section, Subjective section, and Model Papers*. The theory has been explained through solved examples. To enhance problem solving skills of candidates, *Multiple Choice Questions (MCQs)* with detailed solutions are given at the end of each chapter. Two *Mock Test Papers* have been included to understand the pattern of exam. A CD containing Study Chart for systematic preparation, Tips & Tricks to crack Maths Olympiad, Pattern of exam, and links of Previous Years Papers is accompanied with this book. The books are also useful for various competitive exams such as NTSE, NSTSE, and SLSTSE as well.

We wish you all success in the examination and a very bright future in the field of mathematics.

All the best

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Section 1

MATHEMATICAL REASONING

1 Real Numbers

Key Points

Natural Numbers	N = {1, 2, 3, 4, ...}
Whole Numbers	W = {0, 1, 2, 3, 4, ...}
Integers	Z = {. . . . . –3, –2, –1, 0, 1, 2, 3, 4, ...}
Rational Numbers	Q = {$\frac{p}{q}$, $q \neq 0$, p q are co-prime}

These numbers when expressed as decimal representation are either finite or infinite recurring.

Irrational Numbers: Those which are not rational e.g. $\sqrt{3}$, $\sqrt{5}$, ... π, etc.

These numbers when expressed as decimal representation are infinite and non recurring.

Real Numbers: Collection of all above numbers i.e. N, W, Z, Q and irrational numbers.

All numbers can be represented on the number line.

Euclid's Division Algorithm: This is based on Euclid's division lemma.

Euclid's Division lemma: Given positive integers a and b, there exist unique integers q and r such that $a = bq + r$, $0 \leq r < b$

According to this the H.C.F. of any two positive integers a and b with $a > b$ is obtained as follows:

Step 1. Apply the division lemma to find q and r, where $a = bq + r$, $0 \leq r < b$

Step 2. If $r = 0$ then H.C.F. is b. If $r \neq 0$, apply Euclid's lemma to b and r.

Step 3. Continue the process till the remainder is zero. The divisor at this stage will be H.C.F. (a, b). Also H.C.F. (a, b) = H.C.F. (b, r).

Example: If $a = 117$, $b = 45$

By Euclid's division lemma,

$$117 = 45 \times 2 + 27$$
$$a = bd_1 + r_1,$$

where $d_1 = 2$, $r_1 = 27$

We observe that common divisor of $a = 117$ and $b = 45$ are also common divisor of $b = 45$ and $r_1 = 27$ and vice versa

$$45 = 27 \times 1 + 18$$
$$b = d_2 r_1 + r_2 \text{ where } d_2 = 1 \text{ and } r_2 = 18$$

We observe that common divisor of $r_1 = 27$ and $r_2 = 18$ are the common divisor of $b = 45$ and $r_1 = 27$ and vice versa

Applying Euclid's division lemma on

$$r_1 = 27 \text{ and } r_2 = 18$$
$$27 = 18 \times 1 + 9$$
$$r_1 = d_3 r_2 + r_3, \text{ where } d_3 = 1, r_3 = 9$$

$r_2 = 18$ and $r_3 = 9$ are common divisor of $a = 117$ and $b = 45$ and vice versa

$$18 = 9 \times 2 + 0$$

$r_3 = 9$ is a divisor of $r_2 = 18$ and $r_3 = 9$

$\therefore$ H.C.F. = 9

Example 1 Use Euclid's division algorithm to find H.C.F. of 4052 and 12576.

(a) 3 (b) 2 (c) 1 (d) 4

Solution : **(a)** $12576 = 4052 \times 3 + 400$

$4052 = 420 \times 9 + 272$

$420 = 272 \times 1 + 148$

$272 = 148 \times 1 + 124$

$148 = 124 \times 1 + 24$

$124 = 24 \times 5 + 4$

$24 = 4 \times 6 + 0$

Divisor at the last stage or remainder at the earlier stage is the H.C.F i.e, 4.

Example 2 Find the H.C.F. of 65 and 117 and express it in the form $65m + 117n$.

(a) 13 (b) 11 (c) 10 (d) 5

Solution : **(a)**

Here $117 = 65 \times 1 + 52$

$65 = 52 \times 1 + 13$

$52 = 13 \times 4 + 0$

H.C.F. of 65 and 117 = 13

Now $13 = 65 - 52 \times 1$

$13 = 65 - (117 - 65 \times 1)$

$13 = 65 - 117 + 65 \times 1$

$13 = 65 \times 2 + 117\,(-1)$

$13 = 65 \times 2 - 117$

$13 = 65m + 117n$, where $m = 2$, $n = -1$

Example 3 Find the largest number that divides 2053 and 967 and leaves a remainder of 5 and 7 respectively.

(a) 64 (b) 60 (c) 55 (d) 99

Solution : **(a)**

Here $2053 - 5 = 2048$

$967 - 7 = 960$

Required number = H.C.F. of 2048 and 960

```
960) 2048 (2
     1920
     ----
     128) 960 (7
          896
          ---
          64) 128 (2
              128
              ---
```

The largest number = 64

If x and y are two numbers then the product of L.C.M. and H.C.F. is equal to product of these numbers i.e., xy

Example 4 If the H.C.F. of 592 and 252 is 7, then what is their L.C.M.?

(a) 21312 (b) 21000 (c) 21311 (d) 21310

Solution: **(a)**

L.C.M. × H.C.F = 592 × 252

$\Rightarrow$ L.C.M. × 7 = 592 × 252

$$\Rightarrow \quad \text{L.C.M.} = \frac{592 \times 252}{7} = 592 \times 36$$

$\Rightarrow$ = 21312

Fundamental Theorem of Arithmetic

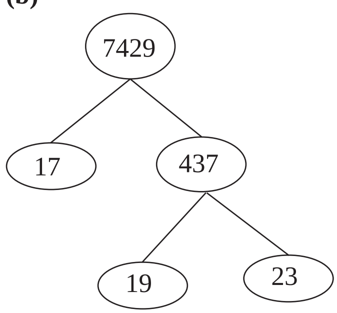
Every composite number can be expressed as a product of primes and this factorization is unique apart from the order in which the prime factors occur.

Revisiting Irrational Numbers: Let p be a prime number. If p divides a^2, then p divides a, where a is a positive integer.

Example 5 Express 7429 as the product of prime factors.

(a) 21 × 17 × 13 (b) 17 × 19 × 23 (c) 51 × 19 × 29 (d) none of these

Solution : **(b)**



7429 = 17 × 19 × 23

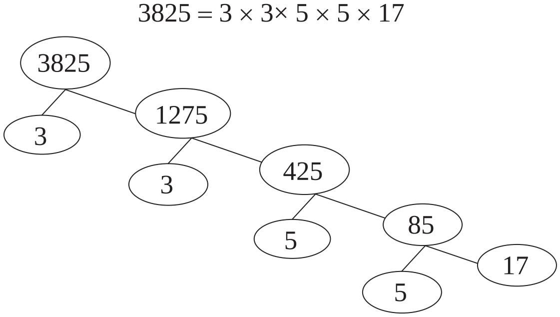
Example 6 Express 3825 as the product of prime factors.

(a) 3 × 3 × 5 × 17 (b) 3 × 3 × 5 × 5 × 17

(c) 3 × 5 × 17 (d) none of these

Solution : **(b)**

3825 = 3 × 3× 5 × 5 × 17



Revisiting Rational Numbers and their Decimal Expansions

I. Let x be a rational number whose decimal expansion terminates. Then x can be expressed in the form $\frac{P}{d}$, where P and d are co–primes. The Prime factorization of d is of the form $2^m \times 5^n$, where m and n are non–negative integers.

Example 7 Find the decimal expansion of $\frac{189}{125}$.

(a) 1.5 (b) 1.4 (c) 1.1 (d) none of these

Solution : **(a)** $\frac{189}{125} = \frac{189}{5^3} = \frac{2^3 \times 189}{2^3 \times 5^3}$

$$= \frac{8 \times 189}{(2 \times 5)^3} = \frac{1512}{10^3} = 1.512$$

Example 8 Find decimal expansion of $\frac{7}{8}$.

(a) 8.5 (b) 0.875 (c) 8.4 (d) 0.878

Solution: **(b)** $\frac{7}{8} = \frac{7}{2^3} = \frac{7 \times 5^3}{2^3 \times 5^3}$

$$= \frac{7 \times 125}{(2 \times 5)^3} = \frac{875}{10^3}$$

$$= 0.875$$

II. Let $x = \frac{P}{d}$ be a rational number such that the prime factorization of d is of the form $2^m \times 5^n$ where m and n are non–negative integers then x has a decimal expansion which terminates after K places of decimals, where K is the larger of m and n.

Example 9 At how many places of decimal $\frac{13}{3125}$ terminates?

(a) 5 (b) 4 (c) 3 (d) 2

Solution: **(a)** $\frac{13}{3125} = \frac{13}{2^0 \times 5^5}$

So, it has terminating decimal expansion which terminates after 5 places of decimal.

III. If $x = \frac{P}{d}$ be a rational number such that the prime factorization of d is of the form $2^m \times 5^n$, where m and n are non-negative integers, then x has a decimal expansion which is non terminating repeating.

Example 10 Is $\frac{64}{455}$ non-terminating repeating?

(a) yes (b) no (c) can't say (d) none of them

Solution : **(a)** $\frac{64}{455} = \frac{64}{5 \times 7 \times 13}$

As 455 is not of the form $2^m \times 5^n$

So, the decimal expansion of $\frac{64}{455}$ is non-terminating repeating

Multiple Choice Questions

1. What is the largest number that divides 445, 572 and 699 having remainders 4, 5, 6 respectively?
(a) 63 (b) 65
(c) 62 (d) 64

2. If the H.C.F. of 408 and 1032 is expressed in the form of $1032m - 408 \times 5$. What is the value of m?
(a) 4 (b) 2 (c) –2 (d) 3

3. What is the largest number which divides 615 and 963 leaving remainder 6 in each case?
(a) 78 (b) 76
(c) 87 (d) 83

4. If 5005 is expressed in the term of product of price factors, then which prime factor is the largest?
(a) 5 (b) 11 (c) 13 (d) 17

5. What is the L.C.M. of 144, 180 and 192 by prime factorization method?
(a) 2680 (b) 2780
(c) 2880 (d) None of these

6. The H.C.F. of two numbers is 16 and their product is 3072. What is their L.C.M.?
(a) 192 (b) 172
(c) 152 (d) 186

7. What is the largest positive integer that will divide 398, 436 and 542 leaving remainder 7, 11 and 15 respectively?
(a) 16 (b) 18
(c) 17 (d) 14

8. The H.C.F of two numbers is 145, their L.C.M. is 2175. If one number is 725, then what is the other number?
(a) 435 (b) 425
(c) 415 (d) 465

9. What is the smallest number that when divided by 35, 56, and 91 leaves remainder 7 in each case?
(a) 3847 (b) 3647
(c) 3247 (d) 3547

10. If L.C.M. and H.C.F. of two rational numbers are equal then the numbers must be
(a) Prime (b) Equal
(c) Co–prime (d) Composite

11. $2 + \sqrt{2}$ is
(a) Irrational (b) an integer
(c) not real (d) rational

12. $1.23\overline{48}$ is
(a) an integer
(b) an irrational number
(c) a rational number
(d) None of these

13. Find the sum of the exponents of the prime factors in the prime factorization of 196.
(a) 2 (b) 3 (c) 4 (d) 5

14. What is the H.C.F of 95 and 152?
(a) 1 (b) 19 (c) 38 (d) 57

15. Find the smallest number by which $\sqrt{27}$ should be multiplied so as to get a rational number.
(a) $\sqrt{27}$ (b) $3\sqrt{3}$
(c) 3 (d) $\sqrt{3}$

16. If n is a natural number then $9^2n - 4^2n$ is always divisible by
(a) 5 (b) 13
(c) both 5 and 13 (d) None of these

17. If n is any natural number then $6^n - 5^n$ always ends with
(a) 3 (b) 1 (c) 7 (d) 5

18. Find L.C.M. of 42 and 63.
(a) 120 (b) 126
(c) 115 (d) 116

19. What is the sum of exponents of the prime factors in the prime factorization of 576?
(a) 6 (b) 8 (c) 7 (d) 5

20. What is the difference of exponents of prime factors in prime factorization of 1225?
(a) 1 (b) 2
(c) 2 (d) None of these

21. Find the least number that is divisible by all the numbers between 1 and 10 both inclusive?
(a) 2520 (b) 2320
(c) 1920 (d) 2720

22. $5-\sqrt{3}$ is

(a) rational (b) irrational

(c) real (d) integer

23. In the prime factorization of 13915 what is difference between largest factor and smallest factor?

(a) 18 (b) 23 (c) 17 (d) 15

24. If the H.C.F. of 210 and 55 is expressible in the form $210 \times 5 + 55y$ then what is the value of y?

(a) 19 (b) –19 (c) 15 (d) –15

25. The decimal expansion of the rational number $\frac{43}{2^4 \times 5^3}$ will terminate after

(a) 4 places (b) 2 places

(c) 3 places (d) 5 places

26. The product of H.C.F. and L.C.M. of the smallest prime number and smallest composite number is

(a) 2 (b) 6 (c) 4 (d) 8

27. Which of the following number has terminating decimal expansion?

(a) $\frac{17}{49}$ (b) $\frac{21}{2^3.5^6}$

(c) $\frac{89}{2^3.3^2}$ (d) $\frac{37}{45}$

28. What is product of H.C.F. and L.C.M. of the numbers 81 and 50?

(a) 900 (b) 4050

(c) 8100 (d) 2100

29. The decimal expansion of $\frac{147}{120}$ will terminate after how many places of decimal?

(a) 1 (b) 2

(c) 3 (d) None of these

30. If H.C.F. of 306 and 657 is 9, then what is the L.C.M. of 306 and 657?

(a) 22338 (b) 22318

(c) 22238 (d) 22118

31. If n is a natural number, then $9^{2n} - 4^{2n}$ is always divisible by ________.

(a) 5

(b) 13

(c) both (a) and (b)

(d) neither (a) nor (b)

32. N is a natural number such that when N^3 is divided by 9, it leaves remainder a. It can be concluded that

(a) a is a perfect square

(b) a is a perfect cube

(c) both (a) and (b)

(d) neither (a) nor (b)

33. If n is any natural number, then $6^n - 5^n$ always ends with ________.

(a) 1 (b) 3 (c) 5 (d) 7

34. Which of the following is always true?

(a) The rationalising factor of a number is unique

(b) The sum of two distinct irrational numbers is rational

(c) The product of two distinct irrational numbers is irrational

(d) none of these

35. Ashok has two vessels which contain 720 ml and 405 ml of milk respectively. Milk in each vessel is poured into glasses of equal capacity to their brim. Find the minimum number of glasses which can be filled with milk.

(a) 45 (b) 35 (c) 25 (d) 30

Answer Key

1. (a)	2. (b)	3. (c)	4. (c)	5. (c)	6. (a)	7. (c)	8. (a)	9. (b)	10 (b)
11. (a)	12. (c)	13. (c)	14. (b)	15. (d)	16. (c)	17. (b)	18. (b)	19. (b)	20. (c)
21. (a)	22. (b)	23. (a)	24. (b)	25. (a)	26. (d)	27. (b)	28. (b)	29. (c)	30. (a)
31. (c)	32. (c)	33. (a)	34. (d)	35. (c)					

Hints *and* Solutions

1. **(a)** Here 445 – 4 = 441; 572 – 5 = 567, 699 – 6 = 693

Now we have to find H.C.F. of 441 , 567, 693.

2. **(b)** Given H.C.F. of 408 and 1032 = 24 and $1032m - 408 \times 5 = 24$

$\Rightarrow \quad 1032m = 24 + 2040$

$\Rightarrow \quad 1032m = 2064$

$\Rightarrow \quad m = \frac{2064}{1032} = 2$

3. **(c)** Here 615 – 6 = 609; 963 – 6 = 957

$\therefore$ H.C.F. of 609 and 957 = 87

4. **(c)** $5005 = 5 \times 7 \times 11 \times 13$

5. **(c)** L.C.M. of 144, 180, 192

$144 = 2^4 \times 3^2$

$180 = 2^2 \times 3^2 \times 5$

$192 = 2^6 \times 3$

$\therefore$ L.C.M. $= 2^6 \times 3^2 \times 5 = 2880$

6. **(a)** L.C.M. $= \frac{\text{product of numbers}}{\text{H.C.F. of numbers}}$

$= \frac{3072}{16} = 192$

7. **(c)** 398 – 7 = 391; 436 – 11 = 425;

542 – 15 = 527

Required number = H.C.F. of 391, 425 and 527 = 17

8. **(a)** Other number $= \frac{\text{H.C.F.} \times \text{L.C.M.}}{\text{one numbers}}$

$= \frac{145 \times 2175}{725} = 435$

9. (b) L.C.M. of 35, 56, 91 = 3640

Remainder = 7

Required number = 3640 + 7 = 3647

13. **(c)** We have $196 = 2^2 \times 7^2$

2 + 2 = 4

14. **(b)**. H.C.F. of 95 and 152 = 19

15. **(d)** $\sqrt{27} \times \sqrt{3} = \sqrt{81} = 9$

17. **(b)** $6^n - 5^n = 6^1 - 5^1 = 1$

$6^2 - 5^2 = 36 - 25 = 11$ and so on

18. **(b)** $324 = 2^2 \times 3^2$

19. **(b)** $576 = 2^6 \times 3^4$

6 + 2 = 8

20. **(c)** Here $1225 = 5^2 \times 7^2$

$\therefore$ Required difference = 2 – 2 = 0

21. **(a)** L.C.M. of 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 = 2520

22. **(b)**

23. **(a)** $13915 = 5 \times 11 \times 11 \times 23$

Required difference = 23 – 5 = 18

24. **(b)** H.C.F. of 210 and 55 = 5

$210 \times 5 + 55y = 5$

$\Rightarrow \quad 55y = 5 - 1050$

$\Rightarrow \quad 55y = -1045$

$\Rightarrow \quad y = \frac{1045}{55} = -19$

25. **(a)** $\frac{43}{2^4 \times 5^3} = \frac{43}{16 \times 125}$

26. **(d)** Smallest prime number = 2

Smallest composite number = 4

$\therefore$ Required product = 2 × 4 = 8

27. **(b)** $\frac{21}{2^3 \times 5^6} = \frac{21}{8 \times 125 \times 125}$

28. **(b)** Required product = H.C.F. × L.C.M.

= 81 × 50 = 4050

30. **(a)** L.C.M. × H.C.F. = 306 × 657

$\Rightarrow$ L.C.M. × 9 = 306 × 657

$\Rightarrow$ L.C.M. $= \frac{306 \times 657}{9} = 22338$

31. **(c)** Given expression is in the form $a^2 - b^2$

$9^{2n} - 4^{2n}$ is divisible by both (9 – 4) and (9 + 4) i.e. 5 and 13.

33. **(a)** For any natural number n, 6^n and 5^n end with 6 and 5 respectively.

35. (c) Here H.C.F. of 720 and 405 is 45

$$
\begin{array}{l}
405\overline{)720}(1 \\
\quad\;\; 405 \\
\quad\;\; \overline{315)405(1} \\
\qquad\quad 315 \\
\qquad\quad \overline{90)315(3} \\
\qquad\qquad 270 \\
\qquad\qquad \overline{45)90(2} \\
\qquad\qquad\quad\;\; 90 \\
\qquad\qquad\quad\;\; \overline{\times}
\end{array}
$$

Hence required number $= \dfrac{720}{45} + \dfrac{405}{45}$

$= \dfrac{1125}{45} = 25$

2 Polynomials

Polynomial

An expression of the form $P(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$ where $a_n \neq 0$ is called a **polynomial** in x of degree n. Here $a_0, a_1, ... a_n$ are real numbers and each power of x is a non-negative integer.

Example:

$2x + 3$ is a polynomial of degree 1.

$x^2 + x + 2$ is a polynomial of degree 2.

Types of Polynomial

There are many types of polynomials.

1. Linear Polynomial

A polynomial of degree one is called a linear polynomial. It is of the form $ax + b$, $a \neq 0$.

Example: $2x + 5, \sqrt{2x - 3}, x - \frac{2}{5}$ etc.

2. Quadratic Polynomial

A polynomial having degree 2 is called a quadratic polynomial.

It is of the form $P(x) = ax^2 + bx + c$, $a \neq 0$.

Example: $x^2 + x + 2, 3x^2 - \sqrt{2x} + 5$ etc.

3. Cubic Polynomial

A polynomial of degree 3 is called a cubic polynomial.

It is of the form $ax^3 + bx^2 + cx + d, a \neq 0$

Example: $x^3 - 2x^2 + 3x + 1, \sqrt{2}z^3 - z^2 + 2z + 5$

Value of a Polynomial at a given Point

If $P(x)$ is a polynomial in x and if α is any real number then the value obtained by putting $x = \alpha$ in $P(x)$ is called the value of $P(x)$ at $x = \alpha$.

Example 1: If $P(x) = 2x^2 - 3x + 5$ then find $P(-5)$ and $P(3)$.

Solution: We have $P(x) = 2x^2 - 3x + 5$

then $P(-5) = 2(-5)^2 - 3(-5) + 5$

$= 50 + 15 + 5 = 70$

and $P(3) = 2(3)^2 - 3(3) + 5$

$= 18 - 9 + 5 = 23 - 9 = 14.$

Zeros of a Polynomial

A real number α is called a zero of the polynomial $P(x)$ if $P(\alpha) = 0$.

Example 2: If $P(x) = x^2 - 2x - 3$, find the zeros of polynomial.

Solution: Given $P(x) = x^2 - 2x - 3$

$$= x^2 - 3x + x - 3$$
$$= x(x - 3) + 1(x - 3)$$
$$= (x + 1)(x - 3)$$

then $P(3) = 3^2 - 2(3) - 3 = 9 - 6 - 3$

$$= 9 - 9 = 0$$

$$P(-1) = (-1)^2 - 2(-1) - 3$$
$$= 1 + 2 - 3 = 0$$

So, 3 and –1 are zeros of the polynomial.

Relation between Zeros and Coefficients of a Quadratic Polynomial

If α and β are the zeros of $P(x) = ax^2 + bx + c,\ a \neq 0$ then

$$\alpha + \beta = \frac{-b}{a}$$

and $\alpha\beta = \dfrac{c}{a}$

A quadratic polynomial whose zeros are α and β is given by $x^2 - (\alpha + \beta)x + \alpha\beta$.

Example 3: Find the zeros of the polynomial.

$f(x) = x^2 + 7x + 12$.

Solution: We have $f(x) = x^2 + 7x + 12$

$$= x^2 + 4x + 3x + 12$$
$$= x(x + 4) + 3(x + 4)$$
$$= (x + 4)(x + 3)$$

$$f(x) = 0$$

$\Rightarrow$ $(x + 4)(x + 3) = 0$

$\Rightarrow$ $x + 4 = 0 \quad \Rightarrow x = -4$

or $x + 3 = 0 \quad \Rightarrow x = -3$

Example 4: Find the sum and product of zeros of the polynomial $2x^2 + 5x - 12$.

Solution: We have $f(x) = 2x^2 + 5x - 12$

then sum of zeros $= \dfrac{-5}{2}$

and product of zeros $= \dfrac{-12}{2} = -6$.

Example 5: Find the quadratic polynomial the sum and product of whose zeros are –5 and 6 respectively.

Solution: Given sum of zeros = –5

and product of zeros = 6

then required polynomial $= x^2 - (-5)x + 6 = x^2 + 5x + 6$.

Example 6: Find the quadratic polynomial whose zeros are $\frac{2}{3}$ and $\frac{-1}{4}$

Solution: Here sum of zeros $= \frac{2}{3} - \frac{1}{4} = \frac{8-3}{12}$

$= \frac{5}{12}$

and product of zeros $= \frac{2}{3} \times \frac{-1}{4} = \frac{-1}{6}$.

Required quadratic polynomial $= x^2 - \left(\frac{5}{12}\right)x + \left(-\frac{1}{6}\right)$

$= x^2 - \frac{5}{12}x - \frac{1}{6} = 12x^2 - 5x - 2.$

Relation between the Zeros and Co-efficients of a Cubic Polynomial

If $P(x) = ax^3 + bx^2 + cx + d$, $a \neq 0$ is a cubic polynomial.

and α, β , γ are the zeros of the polynomial, then

1. $\alpha + \beta + \gamma = \frac{-b}{a}$,
2. $\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$
3. $\alpha\beta\gamma = \frac{-d}{a}$

A cubic polynomial whose zeros are α, β, γ is given by

$x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)x - \alpha\beta\gamma$

Example 7: Find a cubic polynomial whose zeros are –2, –3, and –1.

Solution: The required polynomial is

$[x - (-2)]\ [x - (-3)]\ [x - (-1)]$

$= (x + 2)\ (x + 3)\ (x + 1)$

$= (x^2 + 3x + 2x + 6)\ (x + 1)$

$= (x^2 + 5x + 6)\ (x + 1)$

$= x^3 + x^2 + 5x^2 + 5x + 6x + 6$

$= x^3 + 6x^2 + 11x + 6$

Example 8: Find a cubic polynomial whose zeros are α, β, γ such that $\alpha + \beta + \gamma = 4$, $\alpha\beta + \beta\gamma + \gamma\alpha = 1$, $\alpha\beta\gamma = -6$.

Solution:

$x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)x - \alpha\beta\gamma$

$= x^3 - 4x^2 + x - (-6)$

$= x^3 - 4x^2 + x + 6$

Division Algorithm for Polynomials

Division Algorithm: It states that given any polynomial $p(x)$ and any non-zero polynomial $g(x)$, there are polynomials $q(x)$ and $r(x)$ such that

$p(x) = g(x)\,q(x) + r(x)$

where $r(x) = 0$, or degree $r(x) <$ degree $g(x)$.

Example 9: What is the quotient, when $2x^2 + x - 15$ is divided by $x + 3$?

Solution:

$$
\begin{array}{l}
x+3)\,2x^2 + x - 15\,(2x-5 \\
\quad\ \ \underline{\underset{-}{2x^2} \underset{-}{+} 6x} \\
\qquad\quad -5x - 15 \\
\qquad\quad \underline{\underset{+}{-5x} \underset{+}{-} 15} \\
\qquad\qquad \times
\end{array}
$$

$\therefore$ Quotient $= 2x - 5$

Example 10: What is the dividend if divisor is $x^2 - 2x + 3$, quotient is $5x - 3$ and remainder -5?

Solution:

Dividend = Divisor × quotient + Remainder

$= (x^2 - 2x + 3)(5x - 3) + (-5)$

$= 5x^3 - 3x^2 - 10x^2 + 6x + 15x - 9 - 5$

$= 5x^3 = 13x^2 + 21x - 14$

Multiple Choice Questions

1. What are the zeros of $abx^2 + (b^2 - ac)x - bc$?

(a) $-\frac{b}{a}, \frac{c}{b}$ (b) $\frac{b}{c}, \frac{a}{b}$

(c) $-\frac{b}{c}, \frac{a}{b}$ (d) None of these

2. If a and b are the zeros of $ax^2 + bx + c$, find the value of $\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha}$.

(a) $\frac{3abc - b^3}{a^2c}$ (b) $\frac{b^3 - abc}{ac}$

(c) $\frac{3abc + b^3}{ac^2}$ (d) $\frac{3abc + b^3}{ac^2}$

3. If α, β are the zeros of the polynomial $x^2 - 5x + P$ such that $\alpha - \beta = 1$. What is the value of P?

(a) $\frac{1}{6}$ (b) 6 (c) –6 (d) 3

4. If the sum of squares of zeros of the quadratic polynomial $P(x) = x^2 - 8x + k$ is 40. What is the value of k?

(a) 12 (b) 16 (c) 18 (d) 8

5. If α, β are the zeros of quadratic polynomial $6x^2 + x - 2$, find the value of $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$.

(a) $\frac{12}{25}$ (b) $\frac{-25}{12}$

(c) $\frac{1}{12}$ (d) $\frac{1}{25}$

6. What must be subtracted from $8x^4 + 14x^3 - 2x^2 + 7x - 8$ so that the resulting polynomial is exactly divisible by $4x^2 + 3x - 2$?

(a) $4x - 10$ (b) $14x - 10$

(c) $2x - 10$ (d) $14x + 10$

7. Find the other two zeros of the polynomial $2x^4 - 3x^3 - 3x^2 + 6x - 2$ if its two roots are $\sqrt{2}$ and $-\sqrt{2}$.

(a) 1 and $\frac{1}{2}$ (b) –1 and $\frac{1}{2}$

(c) 2 and 1 (d) –2 and 1

8. What is the cubic polynomial in which the sum, sum of the products of its zeros taken at a time and product of its zeros as 2, –7, –14 respectively?

(a) $k(x^3 - 2x^2 - 7x + 14)$

(b) $k(x^3 - 2x^2 + 8x - 14)$

(c) $k(x^3 + 2x^2 - 7x - 4)$

(d) $k(x^3 - x^2 - x - 14)$

9. Find the zeros of the polynomial $x^3 - 5x^2 - 16x + 80$ if two zeros are equal in magnitude but opposite in sign.

(a) 4, –4, 5 (b) 5, –5, 4
(c) 4, –4, 7 (d) 5, –5, 6

10. If the product of two zeros of the polynomial $2x^3 + 6x^2 - 4x + 9$ is 3 then what is its third zero?

(a) $\frac{3}{2}$ (b) $\frac{3}{2}$ (c) $\frac{1}{2}$ (d) $-\frac{3}{2}$

11. Find a cubic polynomial whose zeros are α, β, γ such that $\alpha + \beta + \gamma = 6$, $\alpha\beta + \beta\gamma + \gamma\alpha = -1$ and $\alpha\beta\gamma = -30$.

(a) $x^3 - 6x^2 - x + 30$

(b) $x^3 + 6x^2 + x - 30$

(c) $x^3 - x^2 - 6x + 30$

(d) None of these

12. If α, β, γ are the zeros of the polynomial $2x^3 + x^2 - 13x + 6$. What is the value of αβγ?

(a) 3 (b) $-\frac{1}{2}$ (c) –3 (d) $-\frac{7}{2}$

13. Find the polynomial which when divided by $-x^2 + x - 1$ gives a quotient $x - 2$ and remainder 3.

(a) $-x^3 + 3x^2 - 3x + 5$

(b) $-x^3 - 3x^2 - 3x - 5$

(c) $x^3 - 3x^2 + 3x - 5$

(d) None of these

14. What must be added to $f(x)$ = $4x^4 + 2x^3 - 2x^2 + x - 1$ so that the resulting polynomial is divisible by $g(x) = x^2 + 2x - 3$?

(a) $61x - 65$ (b) $-61x + 65$
(c) $-61x - 65$ (d) None of these

15. If the polynomial $6x^4 + 8x^3 + 17x^2 + 21x + 7$ is divided by another polynomial $3x^2 + 4x + 1$, the remainder comes out to be $ax + b$. What is the value of a and b?

(a) $a = 1, b = 2$ (b) $a = 2, b = 1$
(c) $a = -2, b = 1$ (d) $a = -1, b = -2$

16. What is the cubic polynomial whose zeros are α, β, γ such that $\alpha + \beta + \gamma = 4$. $\alpha\beta\gamma = -6$ and $\alpha\beta + \beta\gamma + \gamma\alpha = 1$?

(a) $x^3 - 4x^2 + x + 6$

(b) $x^3 - 2x^2 - x + 6$

(c) $x^3 - 4x^2 + 4x - 6$

(d) $x^3 - 4x^2 - x - 6$

17. Find a cubic polynomial whose roots are –2, –3, and –1.

(a) $x^3 - 6x^2 + 9x + 6$

(b) $x^3 + 6x^2 + 11x + 6$

(c) $x^3 + 6x^2 - 11x - 6$

(d) None of these

18. Which quadratic polynomial has sum of whose zeros is –5 and product of its zero is –12?

(a) $x^2 + 5x - 12$ (b) $x^2 - 5x + 12$

(c) $x^2 - 5x + 6$ (d) $x^2 - 10x + 12$

19. If divisor is $2 - x + x^2$ and quotient is $(3x - 1)$ then what is the dividend if dividend is completely divisible?

(a) $3x^3 - 2x^2 - 7x + 2$

(b) $3x^3 - 4x^2 + 7x - 2$

(c) $3x^3 - 4x^2 - 7x + 2$

(d) None of these

20. What are the zeros of $x^2 - 2x - 3$?

(a) 1, –3 (b) –3, –1
(c) 3, 1 (d) 3, –1

21. Find the quadratic polynomial whose zeros are $\frac{2}{3}$ and $-\frac{1}{4}$.

(a) $\frac{1}{12}(4x^2 - 5x - 2)$

(b) $\frac{1}{12}(12x^2 - 5x - 2)$

(c) $\frac{1}{12}(12x^2 - 2x + 5)$

(d) $\frac{1}{12}(12x^2 + 5x - 2)$

22. If α, β are zeros of $2x^2+5x-10$ then what is the value of αβ?

(a) –5 (b) 5

(c) $\frac{2}{5}$ (d) $\frac{-5}{2}$

23. The product of zeros of the polynomial x^3+4x^2+x-6 is

(a) –4 (b) 4 (c) –6 (d) 6

24. If *l*, *m*, *n* are the zeros of polynomial x^3-px^2+dx-r then what is the value of $\frac{1}{lm}+\frac{1}{mn}+\frac{1}{nl}$?

(a) $\frac{p}{r}$ (b) $\frac{r}{p}$

(c) $-\frac{p}{r}$ (d) $\frac{-r}{p}$

25. If one zero of the polynomial $(K^2+4)x^2+13x+4K$ is reciprocal of the other, what is the value of *K*?

(a) 2 (b) –2 (c) 4 (d) –4

26. If α, β are the zeros of a polynomial such α + β = –6 and αβ = –4, then what is the polynomial?

(a) x^2+6x-4 (b) x^2-6x+4

(c) x^2-6x (d) None of these

27. If $f(x)=4\sqrt{3}x^2+5x-2\sqrt{3}$. If α and β be the zeros of the polynomial. What is the product of zeros?

(a) $\frac{1}{2}$ (b) $-\frac{1}{2}$

(c) $\frac{-5}{4}$ (d) $\frac{-5}{4\sqrt{3}}$

28. If divisor is $x-1-x^2$ and dividend is $3x^2-x^3-3x+5$, then what is the remainder?

(a) $x-3$ (b) 3

(c) –3 (d) $x+3$

29. Find the quotient if dividend is $30x^4+11x^3-82x^2-12x+48$ and divisor is $3x^2+2x-4$.

(a) $10x^2+3x-12$ (b) $10x^2-3x-12$

(c) $10x^2-6x-6$ (d) None of these

30. If α, β are zeros of the polynomial x^2-px+d, then what is the value of $\frac{1}{\alpha}+\frac{1}{\beta}$?

(a) $\frac{p}{d}$ (d) $\frac{d}{p}$

(c) 1 (d) $-\frac{p}{d}$

Answer Key

1. (a)	2. (a)	3. (b)	4. (a)	5. (b)	6. (b)	7. (a)	8. (a)	9. (a)	10 (d)
11. (a)	12. (c)	13. (a)	14. (b)	15. (a)	16. (a)	17. (b)	18. (a)	19. (b)	20. (d)
21. (b)	22. (a)	23. (d)	24. (a)	25. (a)	26. (a)	27. (b)	28. (b)	29. (b)	30. (a)

Hints *and* Solutions

1. **(a)** We have $abx^2+(b^2-ac)x-bc$

$$x=\frac{-(b^2-ac)\pm\sqrt{(b^2-ac)^2+4ab^2c}}{2ab}$$

$$x=\frac{ac-b^2\pm\sqrt{(b^2+ac)^2}}{2ab}$$

$$x=\frac{ac-b^2\pm b^2+ac}{2ab}$$

$$x=\frac{ac-b^2+b^2+ac}{2ab};\frac{ac-b^2-b^2-ac}{2ab}$$

$$x=\frac{2ac}{2ab};\frac{-2b^2}{2ab}$$

$$x=\frac{c}{b};\frac{-b}{a}$$

The zeros of the given polynomial are $\frac{c}{b}$ and $-\frac{b}{a}$

2. **(a)** Given α and β are the zeros of

ax^2+bx+c

$$\therefore\ \alpha+\beta=\frac{-b}{a},\ \alpha\beta=\frac{c}{a}$$

$$\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha}=\frac{\alpha^3+\beta^3}{\alpha\beta}$$

$$=\frac{(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)}{\alpha\beta}$$

$$=\frac{\left(\frac{-b}{a}\right)^3-\frac{3c}{a}\left(\frac{-b}{a}\right)}{\frac{c}{a}}$$

$$=\frac{-\frac{b^3}{a^3}+\frac{3bc}{a^2}}{\frac{c}{a}}=\frac{\frac{-b^3+3abc}{a^3}}{\frac{c}{a}}$$

$$=\frac{3abc-b^3}{a^3}\times\frac{a}{c}=\frac{3abc-b^3}{a^2c}$$

3. **(b)** Here α, β are the zeros of x^2-5x+P

then $\alpha+\beta=5$ …(1)

and $\alpha\beta=P$

$\alpha-\beta=1$ …(2)

Solving (1) and (2) we get

$2\alpha=6\ \Rightarrow \alpha=3$

and $\beta=5-3=2$

$\therefore\ \alpha\beta=P=(3)(2)=6$

4. **(a)** $P(x)=x^2-8x+k$

If α, and β are its zeros

then $\alpha+\beta=8,\ \alpha\beta=k$

and $\alpha^2+\beta^2=40$

$\Rightarrow\ (\alpha+\beta)^2-2\alpha\beta=8^2-2k$

$\Rightarrow\ 40=64-2k$

$\Rightarrow\ 2k=24$

$\Rightarrow\ k=12$

5. **(b)** Given $6x^2+x-2$

If α, β are its zeros

then $\alpha+\beta=\frac{-1}{6},\ \alpha\beta=\frac{-2}{6}=-\frac{1}{3}$

Now $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{\alpha^2+\beta^2}{\alpha\beta}$

$$=\frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}$$

$$=\frac{\left(\frac{-1}{6}\right)^2-2\left(\frac{-1}{3}\right)}{-\frac{1}{3}}$$

$$=\frac{\frac{1}{36}+\frac{2}{3}}{\frac{-1}{3}}=\frac{\frac{1+24}{36}}{\frac{-1}{3}}$$

$$=\frac{25}{36}\times\frac{-3}{1}=-\frac{25}{12}$$

6. **(b)** Here

$$4x^2+3x-2\overline{)\,8x^4+14x^3-2x^2+7x-8}(2x^2+2x-1$$
$$\underline{-8x^4 \pm 6x^3 \mp 4x^2}$$
$$8x^3+2x^2+7x$$
$$\underline{-8x^3 \pm 6x^2 \mp 4x}$$
$$-4x^2+11x-8$$
$$\underline{\mp 4x^2 \mp 3x \pm 2}$$
$$14x-10$$

$\therefore$ $14x - 10$ must be subtracted.

7. **(a)** $2x^4-3x^3-3x^2+6x-2$

Let the other roots be α and β.

Sum of the roots $= \sqrt{2}+(-\sqrt{2})+\alpha+\beta = \frac{3}{2}$

$$\alpha+\beta = \frac{3}{2} \qquad \ldots(1)$$

Product of roots $= \frac{-2}{2} = -1$

$$\sqrt{2}+(-\sqrt{2})\alpha\beta = -1 \Rightarrow \alpha\beta = \frac{1}{2} \qquad \ldots(2)$$

$$(\alpha-\beta)^2 = (\alpha+\beta)^2 - 4\alpha\beta$$

$$= \left(\frac{3}{2}\right)^2 = 4\alpha\beta = \frac{9}{4}-\frac{4}{2}=\frac{1}{4}.$$

$$\alpha-\beta = \pm\frac{1}{2} \qquad \ldots(3)$$

From equation (1) and (3)

$$\alpha = 1,\ \beta = \frac{1}{2} \text{ or } \beta = 1,\ \alpha = \frac{1}{2}$$

8. **(a)** Given $\alpha+\beta+\gamma = 2 = \frac{-b}{a} \Rightarrow b = -2a$

$$\alpha\beta+\beta\gamma+\gamma\alpha = -7 = \frac{c}{a} \Rightarrow c = -7a$$

$$\alpha\beta\gamma = -14 = \frac{-d}{a} \Rightarrow d = 14a$$

$b : c : d = -2 : -7 : 14$

Cubic polynomial is

$k(x^3 - 2x^2 - 7x + 14)$

9. **(a)** Let the zeros are $\alpha, -\alpha, \beta$.

$$\therefore\ \alpha+(-\alpha)+\beta = -\left(\frac{-5}{1}\right) = 5$$

$$\Rightarrow \qquad \beta = 5$$

and $(\alpha)(-\alpha)(\beta) = \frac{-80}{1}$

$$\Rightarrow \qquad -\alpha^2\beta = -80 \Rightarrow \alpha^2\beta = 80$$

$$\Rightarrow \qquad \alpha^2 = \frac{80}{\beta} = \frac{80}{5} = 16$$

$$\Rightarrow \qquad \alpha = \pm 4$$

The zeros are 4, –4, 5.

10. **(d)** If α and β are two zeros then $\alpha\beta = 3$

Let third root be γ.

$$\alpha\beta\gamma = \frac{-9}{2}$$

$$\Rightarrow \quad 3\gamma = \frac{-9}{2}$$

$$\Rightarrow \quad \gamma = -\frac{9}{2\times 3} = \frac{-3}{2}$$

11. **(a)** Given $\alpha+\beta+\gamma = 6$

$\alpha\beta+\beta\gamma+\gamma\alpha = -1$

$\alpha\beta\gamma = -30$

The cubic polynomial is

$P(x) = x^3 - (\alpha+\beta+\gamma)x^2 + (\alpha\beta+\beta\gamma+\gamma\alpha)x - \alpha\beta\gamma$

$= x^3 - 6x^2 - x - (-30)$

(substituting the given values)

$= x^3 - 6x^2 - x + 30$

12. **(c)** The given polynomial is

$2x^3+x^2-13x+6$

α, β, γ be its zeros

$$\therefore\ \alpha\beta\gamma = \frac{-d}{a} = \frac{-6}{2} = -3$$

13. **(a)**

Polynomial $= (-x^2+x-1)(x-2)+3$

$= -x^3+2x^2+x^2-2x-x+2+3$

$= -x^3+3x^2-3x+5$