Mathematical Engineering

Alain Bretto

Hypergraph Theory

An Introduction

Springer

Mathematical Engineering

For further volumes:
http://www.springer.com/series/8445

Alain Bretto

Hypergraph Theory

An Introduction

Springer

Alain Bretto
Computer Science Department
Universite de Caen
Caen
France

ISSN 2192-4732 ISSN 2192-4740 (electronic)
ISBN 978-3-319-00079-4 ISBN 978-3-319-00080-0 (eBook)
DOI 10.1007/978-3-319-00080-0
Springer Cham Heidelberg New York Dordrecht London

Library of Congress Control Number: 2013932774

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Do not worry if you have difficulties in math,
I can assure you mine are far greater

Albert Einstein

Preface

Hypergraphs are systems of finite sets and form, probably, the most general concept in discrete mathematics. This branch of mathematics has developed very rapidly during the latter part of the twentieth century, influenced by the advent of computer science. Many theorems on set systems were already known at the beginning of the twentieth century, but these results did not form a mathematical field in itself. It was only in the early 1960s that hypergraphs become an independent theory. Hence, hypergraph theory is a recent theory. It was mostly developed in Hungary and France under the leadership of mathematicians like Paul Erdös, László Lovász, Paul Turán,… but also by C. Berge, for the French school. Originally, developed in France by Claude Berge in 1960, it is a generalization of graph theory. The basic idea consists in considering sets as generalized edges and then in calling hypergraph the family of these edges (hyperedges). As extension of graphs, many results on trees, cycles, coverings, and colorings of hypergraphs will be seen in this book.

Hypergraphs model more general types of relations than graphs do. In the past decades, the theory of hypergraphs has proved to be of a major interest in applications to real-world problems. These mathematical tools can be used to model networks, biology networks, data structures, process scheduling, computations and a variety of other systems where complex relationships between the objects in the system play a dominant role. From a theoretical point of view, hypergraphs allow to generalize certain theorems on graphs, even to replace several theorems on graphs by a single theorem of hypergraphs. For instance, the Berge's weak perfect graph conjecture, which says that a graph is perfect if and only if its complement is perfect, was proved thanks to the concept of normal hypergraph. From a practical point of view, they are now increasingly preferred to graphs.

In this book, we give a general and nonstandard presentation of the theory of hypergraphs, although many paragraphs deal with the traditional elements of this theory.

In Chap. 1, we introduce the basic language of hypergraphs. The last paragraphs are devoted to more original concepts such as entropy of hypergraph. Similarities and kernels are also discussed. This chapter could be useful to engineers and anyone interested in applied science.

Chapter 2 provides the first properties such as the Helly property, the König property, and so on. Standard invariants of hypergraphs are also discussed. In Chap. 3, the classical notions of colorings are addressed.

In the early 1980s, scientists information theory introduced decomposition-join approaches into the design and study of databases with large size. A decomposition of a relation induces a database scheme, that is a hypergraph on the attributer set. So tree and hypertree decompositions are introduced at the end of Chap. 4, as well as the concept of acyclicities which are important in computer science. The first paragraphs are devoted to specific classical hypergraphs. The last paragraph introduces planarity.

With the emergence of information sciences and life science, the sizes of the systems we deal with are becoming bigger and bigger. Hence Chap. 5 is devoted to the reduction of hypergraphs. These reductions make it possible to preserve good topological, combinatorial, and geometrical properties such as connexity, colorings, planarity, and so on. Thus, to solve a problem on hypergraph, we can develop algorithms on its reduced hypergraph.

Chapter 6 deals with directed hypergraphs. We give some of their basic properties, then we study the cycles in a directed hypergraph. We also introduce the notion a algebraic representation of a dirhypergraph.

Chapter 7 gives some applications, not exhaustive and some prospective on hypergraphs.

To summarize, this book can be divided into three, five, seven chapters or levels. Three chapters represent learning, five chapters represent the knowledge of the theory, seven chapters represent the culmination of everything the reader has worked in the three and five levels.

Paris, December 2012 Alain Bretto

Acknowledgments

I make a point of cordially thanking all those who helped me in preparing this book especially:

- Alain Faisant (Associate Professor to Université de Saint-Étienne),
- Francois Hennercart (Professor to Université de Saint-Étienne),
- Thierry Vallée.

Contents

Chapter 1
Hypergraphs: Basic Concepts

VIEW the significant developments of combinatoric thanks to computer science [And89, LW01], hypergraphs are increasingly important in science and engineering. Hypergraphs are a generalization of graphs, hence many of the definitions of graphs carry verbatim to hypergraphs. In this chapter we introduce basic notions about hypergraphs. Most of the vocabulary used in this book is given here and most of this one is a generalization of graphs languages [LvGCWS12].

1.1 First Definitions

A *hypergraph* H denoted by $H = (V; E = (e_i)_{i \in I})$ on a finite set V is a family $(e_i)_{i \in I}$, (I is a finite set of indexes) of subsets of V called *hyperedges*. Sometimes V is denoted by $V(H)$ and E by $E(H)$.
The *order of the hypergraph* $H = (V; E)$ is the cardinality of V, i.e. $|V| = n$; its *size* is the cardinality of E, i.e. $|E| = m$.

By definition the *empty hypergraph* is the hypergraph such that:

- $V = \emptyset$;
- $E = \emptyset$.

Always by definition a *trivial hypergraph* is a hypergraph such that:

- $V \neq \emptyset$;
- $E = \emptyset$.

In the sequel, unless stated otherwise, hypergraphs have a nonempty set of vertices, a non-empty set of hyperedges and they do not contain empty hyperedge.

A. Bretto, *Hypergraph Theory*, Mathematical Engineering,
DOI: 10.1007/978-3-319-00080-0_1,

Let $(e_j)_{j\in J}$, $J \subseteq I$ be a subfamily of hyperedges of $E = (e_i)_{i\in I}$, we denote the set of vertices belonging to $\cup_{j\in J} e_j$ by $V(\cup_{j\in J} e_j)$, but sometimes we use e for $V(e)$. For instance, sometimes we use $e \cap V'$ for $V(e) \cap V'$, $V' \subseteq V$.

If

$$\bigcup_{i\in I} e_i = V$$

the hypergraph is without *isolated vertex*, where a vertex x is isolated if

$$x \in V \setminus \bigcup_{i\in I} e_i.$$

A hyperedge $e \in E$ such that $|e| = 1$ is a *loop*.
Two vertices in a hypergraph are *adjacent* if there is a hyperedge which contains both vertices. In particular, if $\{x\}$ is an hyperedge then x is adjacent to itself. Two hyperedges in a hypergraph are *incident* if their intersection is not empty.

Let $H = (V; E = (e_i)_{i\in I})$ be a hypergraph:

- The *induced subhypergraph* $H(V')$ of the hypergraph H where $V' \subseteq V$ is the hypergraph $H(V') = (V', E')$ defined as

$$E' = \left\{V(e_i) \cap V' \neq \emptyset : e_i \in E \text{ and either } e_i \text{ is a loop or } |V(e_i) \cap V'| \geq 2\right\}$$

  The letter E' can be represented a multi-set. Moreover, according to the remark above we can add, if we need, the emptyset.
- Given a subset $V' \subseteq V$, the *subhypergraph* H' is the hypergraph

$$H' = (V', E' = (e_j)_{j\in J}) \text{ such that for all } e_j \in E' : e_j \subseteq V';$$

- A *partial hypergraph* generated by $J \subseteq I$, H' of H is a hypergraph

$$H' = (V', \left(e_j\right)_{j\in J}).$$

  where $\bigcup_{j\in J} e_j \subseteq V'$. Notice that we may have $V' = V$.

The *star* $H(x)$ centered in x is the family of hyperedges $(e_j)_{j\in J}$ containing x; $d(x) = |J|$ is the *degree* of x excepted for a loop $\{x\}$ where the degree $d(x) = 2$. If the hypergraph is without repeated hyperedge the degree is denoted by $d(x) = |H(x)|$, excepted for a loop $\{x\}$ where the degree $d(x) = 2$. The maximal degree of a hypergraph H is denoted by $\Delta(H)$.

If each vertex has the same degree, we say that the hypergraph is *regular*, or k-regular if for every $x \in V$, $d(x) = k$.